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LETTER TO THE EDITOR

Dynamical localization for two-level systems periodically driven

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Abstract

Here, we consider a two-level system driven by an external periodic field. We show that the coherent destruction of tunnelling, as proved by Grossmann and co-workers (1991 *Phys. Rev. Lett.* **67** 516; 1992 *Europhys. Lett.* **18** 571) in the case of a monochromatic field, also appears for any periodic driving field given by an even regular function with zero mean value and satisfying a technical condition on the zeros of this function.

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1. Introduction

Driven two-level systems have been the subject of great interest since the works by Rabi who solved the problem of a two-level spin system in a circularly polarized magnetic field [2]. At present they appear in many fields, from theoretical physics to practical optics (see [3] and [8] and the references therein). For instance, if one considers a one-dimensional quantum system with symmetric potential and an external time-dependent field (see [5, 6])

$$\mathrm{i}\frac{\partial}{\partial t}u(x,t) = -\frac{\partial^2}{\partial x^2}u(x,t) + [x^4 - \beta x^2]u(x,t) + Sxf(\omega t)u(x,t), \qquad S, \beta > 0, \tag{1}$$

it is well known that the autonomous Hamiltonian has two parity-even and -odd eigenstates $v_\pm$ with eigenvalues $\lambda_\pm$. The restriction of equation (1) to the bi-dimensional space spanned by the two eigenvectors $v_\pm$ is usually called the *two-level system* and, in a suitable base, it takes the following form:

$$\mathrm{i}\dot{\phi} = H_1\phi, \qquad H_1 = \epsilon\sigma_1 + \eta f(\omega t)\sigma_3, \qquad \phi(0) = \phi^0, \tag{2}$$

where $\epsilon = \frac{1}{2}|\lambda_+ - \lambda_-| > 0$, η is the real-valued parameter directly proportional to the field's strength, $\dot{\phi}$ denotes the derivative of ϕ with respect to the time t,

$$\phi(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix},$$

ω is the driving frequency, $f(t)$ is a periodic function with period 2π and $\sigma_{1,3}$ are the two Pauli's matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If the external field is absent then $\eta = 0$ and equation (2) takes the form

$$\mathrm{i}\dot{\phi} = \epsilon\sigma_1\phi$$

with solution given by

$$\phi_1(t) = \phi_1(0)\cos(\epsilon t) - \mathrm{i}\phi_2(0)\sin(\epsilon t)$$

and

$$\phi_2(t) = \phi_1(0)\sin(\epsilon t) - \mathrm{i}\phi_2(0)\cos(\epsilon t).$$

Hence $\phi(t)$ is a periodic function with period $\frac{2\pi}{\epsilon}$ and the wavefunction $u(x, t)$ shows a *beating motion* between the two wells.

When we restore the driving field, one of the main addresses concerns the effect of the coherent destruction of the tunnelling (also called dynamical localization). In a seminal paper, Grossmann and co-workers [5, 6] pointed out that the beating motion in a two-level system can be controlled, and even suppressed, by means of a tailored external monochromatic driving field (9).

In this paper, we show that such an effect, that is the suppression of the beating motion for critical value of the parameters, also appears for any even and periodic driven field $f(\omega t)$ satisfying assumptions H1 and H2. More precisely, we are able to prove that (see the theorem below) the integral $\hat{I}$ defined in equation (6) is exactly zero for a suitable choice of the external field's parameters; from this fact and by means of the averaging theorem the destruction of the beating motion follows.

2. Notations and main results

Here, we consider the two-level equation (2) where the actual semiclassical parameter is the beating frequency ϵ.

For our purposes, it will be useful to write the original equation (2) in a different form by means of the transformation

$$\psi = \mathrm{e}^{\mathrm{i}\alpha\sigma_3}\phi$$

where

$$\alpha(t) = \int_0^t \eta f(\omega\xi)\,\mathrm{d}\xi. \tag{3}$$

Then equation (2) takes the form

$$\mathrm{i}\dot{\psi} = H_2\psi, \qquad H_2 = \epsilon\,\mathrm{e}^{\mathrm{i}\alpha\sigma_3}\sigma_1\,\mathrm{e}^{-\mathrm{i}\alpha\sigma_3}, \tag{4}$$

with the same initial condition $\psi(0) = \psi^0 = \phi^0$.

When the driving field is absent, that is $\eta = 0$, then equation (4) has a periodic solution with period

$$T = \frac{2\pi}{\epsilon}.$$

By means of the averaging theorem [9], in the limit of small beating frequency, that is $\epsilon \ll \omega$, and for times of the order of the beating period T we can approximate the solution of equation (4) by the solution of the average system given by

$$\mathrm{i}\dot{\hat{\psi}} = \hat{H}_2\hat{\psi}, \qquad \hat{H}_2 = \epsilon \hat{I}\sigma_1, \tag{5}$$

where

$$\hat{I} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{e}^{2\mathrm{i}\alpha(t)}\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{2\mathrm{i}\chi\int_0^t f(q)\mathrm{d}q}\,\mathrm{d}t, \qquad \chi = \frac{\eta}{\omega}. \tag{6}$$

That is, the unperturbed solution $\psi(t)$ is approximated by means of the solution $\hat{\psi}$ related to the averaged equation (5) for any time of the order $1/\epsilon$: for any $\delta > 0$ there exists $\epsilon_0 > 0$ such that for any ϵ, $0 < \epsilon < \epsilon_0$, then

$$|\psi(t) - \hat{\psi}(t)| < \delta, \qquad \forall t \in [0, T]. \tag{7}$$

It has been found that for a monochromatic driving force (see equation (9)) the wavefunction ϕ is, for certain values of the field's parameters, nearly 'frozen' in its initial configuration [5, 6]; that is we have the dynamical localization effect as defined below.

Definition. *Let ψ be the solution of equation (4) with initial condition ψ^0, let $T = \frac{2\pi}{\epsilon}$ be the unperturbed beating period. The dynamical localization effect, also called coherent destruction of the tunnelling, means that for any $\nu > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ then*

$$|\psi(t) - \psi^0|^2 < \nu, \qquad \forall t \in [0, T]. \tag{8}$$

From the average theorem it follows that we have dynamical localization if, and only if, $\hat{I} = 0$ (actually, in [7], a general criterion has been given for dynamical localization which also holds for non-periodic fields); furthermore, $\hat{I}$ depends only on the ratio $\chi = \eta/\omega$ between the two field's parameters. Therefore, if the monochromatic field has the form

$$f(t) = \tfrac{1}{2}\sin(t), \tag{9}$$

where η and ω are, respectively, the amplitude and the frequency of the external monochromatic field then [1]

$$\hat{I} = J_0(\chi),$$

where $J_0(x)$ is the zeroth Bessel function. From this fact and since (5) has the same form of equation (2) with $\eta = 0$ and ϵ replaced by $\epsilon J_0(\chi)$ it follows that when the external field's parameters η and ω are such that $J_0(\chi) = 0$ then the beating motion disappears and we have dynamical localization.

Now, we show that such an effect generically occurs for a given periodic driven field f provided that $f(t)$ is an even and regular function. More precisely, we assume that

Hypothesis I. *Let $f(t)$ be a real-valued periodic function such that:*

(i) *$f(t)$ is an analytic function on a complex strip* $\mathrm{R} \times \mathrm{i}(-\delta, +\delta)$ *for some $\delta > 0$;*
(ii) *there exists t_0 such that $f(t_0 - t) = f(t_0 + t)$ for any t;*
(iii) *$f(t)$ has mean value zero: $\frac{1}{2\pi}\int_0^{2\pi} f(t)\,\mathrm{d}t = 0$;*
(iv) *$f(t)$ satisfies to the normalization condition $\int_0^{2\pi} f^2(t)\,\mathrm{d}t = 1$.*

In the following let us assume, for the sake of definiteness, that $t_0 = 0$.

Now, let

$$\Omega_n = \{t \in [0, \pi] : f(t) = \dot{f}(t) = \cdots = f^{(n-1)}(t) = 0, f^{(n)}(t) \neq 0\}, \qquad n = 1, 2, \ldots.$$

From assumptions I(ii) and (iii) it follows that there exists at least one $t \in [0, \pi]$ such that $f(t) = 0$, from this fact and from the analyticity of $f(t)$ it follows that such a zero t has a finite multiplicity (otherwise $f(t) \equiv 0$ in contradiction with assumption I(iv)). Hence, there exists $n \geqslant 1$ such that $\Omega_n \neq \emptyset$; let

$$N = \min(n : \Omega_n \neq \emptyset) + 1.$$

From the analyticity of the function $f(t)$ it follows that the set Ω_N has finite cardinality; thus

$$\Omega_N = \{t_j, j = 1, \ldots, m\}, \qquad 0 \leqslant t_1 < t_2 < \cdots < t_m \leqslant \pi,$$

for some $m \geqslant 1$.

Hypothesis II. *Let*

$$\gamma_j = 2 \int_0^{t_j} f(q)\, \mathrm{d}q, \tag{10}$$

we assume that $\gamma_j \neq 0$ *for any* $j = 1, 2, \ldots, m$.

We can now state our main result.

Theorem. *Let hypotheses I and II be satisfied. Then, there exists a critical value of the parameter* χ *such that* $\hat{I}(\chi) = 0$. *That is, we have dynamical localization for critical values of the field's parameters.*

Remark 1. Let us underline that assumption I(i) could be easily weakened; for instance, it is enough to assume that $f(t) \in C^\infty$ has finitely many zeros with finite multiplicity.

Remark 2. Assumption I(iv) is not really necessary, we just normalize f for the sake of definiteness.

Remark 3. Assumptions I(ii) and (iii) are actually crucial. Indeed, if the mean value of f is not zero then the driven field does not give the dynamical localization effect for any frequency ω, but only for some resonance values (see section 4.2 in [7]). For what concerns assumption I(ii) we now show an example of a driven field that does not satisfy such a condition and for which the dynamical localization effect does not hold. To this end we consider the following piecewise function:

$$f(t) = \begin{cases} 1/a & 0 \leqslant t < a \\ -1/(\pi - a) & a \leqslant t < \pi \\ 0 & \pi \leqslant t < 2\pi, \end{cases}$$

where $a \neq 0$ is given (actually the regularity condition I.i) does not hold; however this point is not really crucial). In such a case, a simple computation gives

$$\int_0^t f(q)\, \mathrm{d}q = \begin{cases} t/a & 0 \leqslant t < a \\ (\pi - t)/(\pi - a) & a \leqslant t < \pi \\ 0 & \pi \leqslant t < 2\pi \end{cases}$$

and

$$\hat{I} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{2\mathrm{i}\chi \int_0^t f(q)\mathrm{d}q}\, \mathrm{d}t = \frac{1}{4\chi}[2\chi + \sin(2\chi)] + \mathrm{i}\left[1 - \frac{\cos(2\chi)}{\chi}\right]$$

whose real part is different from zero for any χ since $\sin(x) < x$ for any $x > 0$.

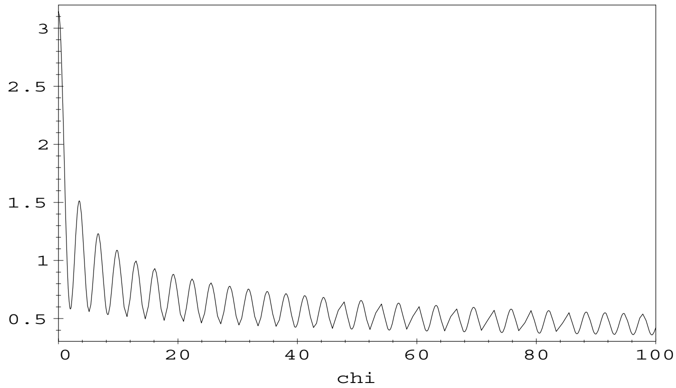

Figure 1. We plot the function $\hat{I}(\chi)$ for the driving field $f(t) = [1 - \cos(2t)]\sin(|2t|)$. In such a case it follows that $\hat{I}(\chi) \neq 0$ for any χ; hence, the phenomenon of the tunnelling destruction does not appear.

Remark 4. If hypotheses II is not satisfied then the theorem fails. For instance, let $f(t) = [1 - \cos(2t)]\sin(|2t|)$ with $d = 2\pi$ (actually this function is not analytic at $t = 0$; however this fact is not really crucial), in such a case we have that $t_1 = 0$, $t_2 = \frac{1}{2}\pi$ and $t_3 = \pi$ with $\gamma_1 = \gamma_3 = 0$ and $\gamma_2 = 1$. Then, in such a case it follows that

$$\hat{I}(\chi) = \int_0^\pi \cos[2\chi(\cos^2(t) - 1)^2]\,\mathrm{d}t$$

and that $\hat{I}(\chi) \neq 0$ for any χ (see figure 1).

3. Proof of the theorem

In order to prove the theorem we give the following preliminary result.

Lemma 1. *We have that*

$$\hat{I}(\chi) = \chi^{-1/N} g(\chi)[1 + o(1)], \qquad \textit{as} \quad \chi \to +\infty,$$

where

$$g(\chi) = \sum_{j=1}^{m} \alpha_j \cos(\gamma_j \chi + \varphi_j) \tag{11}$$

with γ_j defined by (10), for some $N \geqslant 2, m \geqslant 1$ and for given parameters $\alpha_j \neq 0$ and φ_j, $j = 1, 2, \ldots, m$.

Proof. Since $f(t)$ is an even periodic function it follows that:

$$\begin{aligned}
\hat{I} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{2\mathrm{i}\chi \int_0^t f(q)\mathrm{d}q}\,\mathrm{d}t \\
&= \frac{1}{2\pi} \left[\int_{-\pi}^{0} \mathrm{e}^{2\mathrm{i}\chi \int_0^t f(q)\mathrm{d}q}\,\mathrm{d}t + \int_0^{\pi} \mathrm{e}^{2\mathrm{i}\chi \int_0^t f(q)\mathrm{d}q}\,\mathrm{d}t \right] \\
&= \frac{1}{2\pi} \left[\int_0^{\pi} \mathrm{e}^{-2\mathrm{i}\chi \int_0^t f(q)\mathrm{d}q}\,\mathrm{d}t + \int_0^{\pi} \mathrm{e}^{2\mathrm{i}\chi \int_0^t f(q)\mathrm{d}q}\,\mathrm{d}t \right] \\
&= \frac{1}{\pi} \Re \left[\int_0^{\pi} \mathrm{e}^{2\mathrm{i}\chi \int_0^t f(q)\mathrm{d}q}\,\mathrm{d}t \right].
\end{aligned}$$

The stationary phase method (see equation (A.1) in the appendix) gives

$$\int_0^{\pi} \mathrm{e}^{2\mathrm{i}\chi\int_0^t f(q)\,\mathrm{d}q}\,\mathrm{d}t = \chi^{-1/N}h(\chi),$$

where

$$h(\chi) = \frac{2}{N}\Gamma\left(\frac{1}{N}\right)\left(\frac{N!}{2}\right)^{1/N}\sum_{j=1}^{m} c_j|f^{(N-1)}(t_j)|^{-1/N}$$

$$\times\exp\left[\mathrm{i}\frac{\pi}{2N}\delta_j + 2\mathrm{i}\chi\int_0^{t_j} f(q)\,\mathrm{d}q\right][1+o(1)], \qquad \text{as} \quad \chi\to+\infty,$$

and

$$\delta_j = \operatorname{sgn}[f^{(N-1)}(t_j)] \qquad \text{and} \qquad c_j = \begin{cases} 1 & \text{if} \quad t_j\in(0,\pi) \\ \frac{1}{2} & \text{if} \quad t_j = 0,\pi. \end{cases}$$

Hence,

$$\hat{I} = \chi^{-1/N}g(\chi)[1+o(1)], \qquad \text{as} \quad \chi\to+\infty,$$

where $g(\chi)$ is given by (11) with γ_j given by (10) and

$$\varphi_j = \frac{\pi}{2N}\delta_j, \qquad \alpha_j = \frac{2}{N\pi}\Gamma\left(\frac{1}{N}\right)\left(\frac{N!}{2}\right)^{1/N}\sum_{j=1}^{m} c_j|f^{(N)}(t_j)|^{-1/N}. \qquad \square$$

Now, we consider the equation $g(\chi) = 0$ where we prove that it has arbitrarily large solutions. More precisely, we prove that

Lemma 2. *Let $g(\chi) = \sum_{j=1}^m \alpha_j\cos(\gamma_j\chi+\varphi_j)$, where $\alpha_j, \gamma_j \neq 0$, $j = 1, 2, \ldots, m$, then for any $\zeta > 0$ there exists $\chi > \zeta$ and $\chi^{\pm} > \zeta$ such that $g(\chi) = 0$ and $\pm g(\chi^{\pm}) > C$ for some $C > 0$ independent of ζ.*

Proof. Let

$$\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$$

and we set $\Gamma = \Gamma_1 \cup \Gamma_1^\star$, where $\Gamma_1\cap\Gamma_1^\star = \emptyset$ where

$$\Gamma_1 = \left\{\gamma\in\Gamma : \frac{\gamma}{\gamma_1}\in \mathrm{Q}\right\} \qquad \text{and} \qquad \Gamma_1^\star = \left\{\gamma\in\Gamma : \frac{\gamma}{\gamma_1}\in \mathrm{R}-\mathrm{Q}\right\}.$$

Let now $g(\chi) = g_1(\chi) + g_1^\star(\chi)$ where

$$g_1(\chi) = \sum_{j=1,..,m;\gamma_j\in\Gamma_1}\alpha_j\cos(\gamma_j\chi+\varphi_j)$$

and

$$g_1^\star(\chi) = \sum_{j=1,..,m;\gamma_j\in\Gamma_1^\star}\alpha_j\cos(\gamma_j\chi+\varphi_j).$$

By definition, it follows that $g_1(\chi)$ is a periodic function with period $\tilde{T}$, where

$$\tilde{T} = \min\left(T : \frac{T\gamma_j}{2\pi} = n_j\in\mathrm{N}, \gamma_j\in\Gamma_1\right),$$

furthermore

$$\int_0^{\tilde{T}} g_1(\chi)\,\mathrm{d}\chi = \sum_{j=1,..,m;\gamma_j\in\Gamma_1}\int_0^{n_j2\pi/\gamma_j}\alpha_j\cos(\gamma_j\chi+\varphi_j)\,\mathrm{d}\chi = 0.$$

Hence, there exist $\hat{\chi}$, $\hat{\chi}^{\pm} \in [0, T)$ such that

$$g(\hat{\chi}) = 0 \qquad \text{and} \qquad \pm g(\hat{\chi}^{\pm}) > C$$

for some $C > 0$. Now let

$$\chi_\ell = \hat{\chi} + \ell\tilde{T} \qquad \text{and} \qquad \chi_\ell^{\pm} = \hat{\chi}^{\pm} + \ell\tilde{T}, \quad \ell \in \mathbb{Z}.$$

If $\Gamma_1^\star = \emptyset$ the proof is completed. If not, we recall that the frequencies of $\Gamma_1^\star$ are incommensurate with the frequencies of Γ_1. Hence, for any $\epsilon > 0$ we can extract a subsequence $\hat{\chi}_{\ell_r}^{\pm}$ from $\hat{\chi}_\ell^{\pm}$ such that

$$\left|\alpha_j \cos\left(\gamma_j \hat{\chi}_{\ell_r}^{\pm} + \varphi_j\right)\right| < \epsilon, \forall r, \forall j : \gamma_j \in \Gamma_1^\star.$$

In particular, if we choose

$$\epsilon = \min\left(\frac{|g_1(\hat{\chi}^{\pm})|}{2m}\right)$$

then it follows that:

$$\begin{aligned} g\left(\hat{\chi}_{\ell_r}^{+}\right) &= g_1(\hat{\chi}^{+}) + \sum_{j=1,..,m;\gamma_j\in\Gamma_1^\star} \alpha_j \cos\left(\gamma_j \hat{\chi}_{\ell_r}^{+} + \varphi_j\right) \\ &\geqslant g_1(\hat{\chi}^{+}) - m\epsilon > \frac{1}{2}C \end{aligned}$$

and

$$\begin{aligned} g(\hat{\chi}_{\ell_r}^{-}) &= g_1(\hat{\chi}^{-}) + \sum_{j=1,..,m;\gamma_j\in\Gamma_1^\star} \alpha_j \cos\left(\gamma_j \hat{\chi}_{\ell_r}^{-} + \varphi_j\right) \\ &\leqslant g_1(\hat{\chi}^{-}) + m\epsilon < -\frac{1}{2}C. \end{aligned}$$

The proof is complete since, by means of a continuity argument, some χ between $\chi_{\ell_r}^{-}$ and $\chi_{\ell_r}^{+}$ such that $g(\chi) = 0$ exists. □

We can conclude the proof of the theorem. Indeed $\hat{I}$ is a real-valued continuous function such that $\pm\hat{I}(\chi^{\pm}) > 0$ provided that $\chi^{\pm}$ are large enough. From this fact a continuity argument implies that $\hat{I}(\chi) = 0$ for some χ between χ^{-} and χ^{+}.

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Appendix. The stationary phase formula

Here, we recall the following result for the asymptotic expansion of the integral:

$$I = \int_a^b e^{i\lambda\phi(t)} f(t)\, dt, \qquad \text{as} \quad \lambda \to +\infty,$$

where f and ϕ are smooth functions, $a < b$ are given and

$$\dot{\phi}(c) = \cdots = \phi^{(n-1)}(c) = 0, \qquad \phi^{(n)}(c) \neq 0, \qquad n \geqslant 2, \qquad \dot{\phi}(t) \neq 0,$$
$$t \in [a, b], \qquad t \neq c,$$

for some $c \in (a, b)$ and $f(c) \neq 0$. Then (see equation (6.1.12) in [4])

$$I = \frac{2f(c)\Gamma(1/n)}{n}\left[\frac{\lambda|\phi^{(n)}(c)|}{n!}\right]^{-1/n} \exp\{i\lambda\phi(c) + i\,\mathrm{sgn}[\phi^{(n)}(c)]\pi/2n\}[1 + o(1)],$$
$$\text{as} \quad \lambda \to +\infty. \tag{A.1}$$

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